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SIMILARITY ANALYSIS OF INELASTIC CONTACT

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Abstract—Analysis of mechanical contact of solids is of interest not only regarding a variety of mechanical assemblies but also on a smaller scale such as roughness properties of surfaces and compaction of powder particles. Indentation testing is another prominent problem in the context. To analyse the phenomena involved is inherently difficult at application essentially due to the presence of large strains, nonlinear material behaviour, time dependence and moving contact boundaries. Recently, progress has been made, however, to explicitly solve basic boundary value problems especially due to advances in computational techniques. A substantial ingredient which facilitates solution procedures is self-similarity and it is the present purpose to explore in detail the advantages in a general setting when this feature prevails. A viscoplastic framework is laid down for a wide class of constitutive properties where strain-hardening plasticity, creep and also nonlinear elasticity arise as special cases. It is then shown that when surface shapes and material properties are modelled by homogeneous functions, associated boundary value problems posed may be reduced to stationary ones. As a consequence, within Hertzian kinematics, relations between contact impression and regions become independent of loading and time and the connection to loading characteristics does not usually require a full solution of the problem. In particular it is shown that for general head-shapes it proves efficient to use an approach where an intermediate flat die solution serves as a basic tool also for hereditary materials. An invariant computational procedure based on the intermediate problem is arrived at and decisive results shown to be found by simple cumulative superposition. Illustrations are given analytically for ellipsoidal contact of Newtonian fluids and by detailed computations for spherical indentation of viscoplastic solids for which also universal hardness formulae are proposed. For several bodies in contact it is shown how general results may be extracted from fundamental solutions for a half-space. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

Mechanical problems of two bodies in mutual contact seem to have their origin in the analysis of linear elastic solids by Hertz (1882), and have since been of a central nature in the mechanics of solids. Elasticity theory of contacts, one branch of which is often called Hertz theory, has, after the fundamental contribution laid down, made continuous progress accompanied by developments of mathematical techniques based on complex variables, integral transforms and Green functions. Many of the essential results have been summarized, e.g., by Galin (1953), Gladwell (1980) and Hills *et al.* (1993). More recent advances contain not only linear elastic contact theories but also contributions to inelastic behaviour of bodies, Johnson (1985). Such issues include permanent as well as time-dependent deformations and stresses due to contact and are of importance, e.g., at analysis of surface roughness and powder compaction. Indentation tests are also pertinent problems as they aim at exploration of mechanical properties of materials by probing a local region of their surface.

Notwithstanding their importance, complexities in inelastic contact phenomena inevitably make their analytical assessment highly intractable. Several sources of nonlinearity are present in the problems at hand like plastic or viscous material behaviour, moving contact boundaries and frictional effects. Some of these issues also imply that the results are history dependent and thus have to be traced incrementally.

Analytical solutions to plastic contact problems are essentially confined to slip line theories of rigid-perfectly plastic solids with simple geometries c.f., e.g., Hill (1950). Driven by the need for further understanding of this complicated field of mechanics, however, more general means like finite element methods have been ambitiously applied in particular

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to analyse indentation problems. To this end, perhaps, Akyuz and Merwin (1968), Hardy *et al.* (1971) and Lee *et al.* (1972) were the first to analyse indentation of elastic–plastic solids under plane and axisymmetric conditions, respectively. Such numerical achievements have had many followers and become more mature due to the progress of computational techniques when aiming at high accuracy solutions such as recently by Edlinger *et al.* (1993) and Kral *et al.* (1993) for indentation by spheres of strain-hardening elastic–plastic solids. The nonlinearities present still make it a rather formidable procedure to assess accuracy with confidence in nontrivial situations and to condense results with some generality.

When formulating contact problems with emphasis on generality it has proved advantageous to draw upon similarity principles first to clarify the dependence of solutions on governing parameters and more recently to exploit the use of computational techniques more efficiently. In the context of linear elasticity similarity aspects have been explored for a long time cf., e.g., Mossakovski (1963), Spence (1968, 1975), Hill and Storåkers (1990). When it comes to nonlinear material properties, Hill *et al.* (1989) gave a background to empirical hardness formulae and proved for Brinell indentation of power law solids that self-similarity may be applied and as a consequence that the problem of a moving boundary may be reduced to a stationary one. With this as a basis, Hill *et al.* (1989) analysed the problem in depth and besides providing a theoretical interpretation of earlier empirical findings gave a full account of explicit results with the aid of a specially designed finite element procedure. This investigation was, however, based on a material model with no inherent history dependence, i.e., nonlinear elasticity or alternatively deformation theory of plasticity. To achieve generality, though, it is imperative that the incremental behaviour of materials is considered at indentation. To this end, and also to devise efficient computational techniques, an alternative procedure to use an intermediate flat die field followed by cumulative superposition to analyse indentation of nonlinear solids by curved dies was proposed for power law creep by Storåkers and Larsson (1994). Although the idea is an old one in case of linear elasticity, possibly originating from Mossakovski (1963), apparently the approach had never been tried for nonlinear solids where ordinarily superposition principles fail to apply. The technique proved to be efficient, however, and was beneficially employed in full to obtain highly accurate solutions at Brinell indentation also for strain-hardening plastic solids by Biwa and Storåkers (1995).

When fully inelastic behaviour is at issue the similarity strategy sketched can be a strong alternative to computational methods applied by brute force when a moving boundary and natural time have to be considered simultaneously. In their analysis of spherical indentation Biwa and Storåkers (1995) considered as an alternative an elastic–plastic procedure computationally based on the commercial code ABAQUS (1992) in order to obtain asymptotic results in the fully plastic range. Besides the labour involved, several sources of inaccuracy evolved, in particular regarding imprecise determination of the moving contact boundary. The matter has, however, been further pursued in this spirit by Ogbonna *et al.* (1995) where asymptotic states have been sought in a more general case. The investigation by Ogbonna *et al.* (1995) was preceeded by a similarity method proposed by Hill (1992) to analyse spherical indentation of power law creep being reduced to a stationary problem. Hill's procedure was based on total displacements and no explicit results were demonstrated. Bower *et al.* (1993) subsequently transformed the procedure by Hill (1992) to return to an intermediate stationary rate problem and combined it with a finite element code, ABAQUS (1992), employing natural time as an essential variable. In the more general case of viscoplasticity Ogbonna *et al.* (1995) abandoned stationarity altogether and used a combined strategy involving both an evolving contact region and natural time.

The aim here is to examine in detail general features of self-similarity and offer efficient solution procedures at contact of nonlinear solids including natural time though reduced to stationarity. The analysis will first be focused on viscoplastic behaviour in a rate formulation from which classical models for plasticity and creep will emerge as special cases. Introduction of appropriate similarity scaling to allow reduction to a fixed boundary will result in a transformation from material history (time) dependence to spatial nonlocality. Desired solutions may then be obtained from flat die fields followed by cumulative superposition. To this end first a general constitutive background to inelastic material behaviour is briefly summarized and a general theory to analyse contact or indentation is formulated with its particular advantages discussed. A user-friendly computational procedure is outlined in detail and shown to generate results of high accuracy. The proposed methodology is first illustrated by pure analysis of ellipsoidal contact of Newtonian fluids. Detailed computational results are then obtained for spherical indentation of viscoplastic solids and in particular some universal hardness formulae are proposed. Finally it is shown how results may be extended to apply to cases when several deforming bodies of contact are involved.

2. PRELIMINARIES FROM THE THEORY OF INELASTIC SOLIDS

With generality in mind some constitutive equations in the mechanical theory of viscoplastic solids are first laid down where the background essentially originates from articles by Hill (1956, 1987a, 1987b), Rice (1970) and Mroz (1973). In what follows, for simplicity though not by necessity, only solids obeying an associated flow rule are considered as the analysis to follow may be carried out accordingly for non-associated cases as well when suitable homogeneity properties apply. First, two viscoplastic potentials, Φ and Ψ, are introduced, which generate the stress and strain rate as

$$\dot{\varepsilon}_{ij} = \frac{\partial \Phi}{\partial \sigma_{ij}}, \quad \sigma_{ij} = \frac{\partial \Psi}{\partial \dot{\varepsilon}_{ij}}, \tag{1}$$

respectively.

The potentials are related to each other through a dissipation function

$$\mathscr{D} = \sigma_{ij}\dot{\varepsilon}_{ij} = \Phi + \Psi \tag{2}$$

by a Legendre transformation. Thus, if it is assumed that $\Phi(\sigma_{ij})$ is a homogeneous function of degree $(n+1)$, say, then $\Psi(\dot{\varepsilon}_{ij})$ is necessarily homogeneous of degree $(n+1)/n$ due to the above duality nature. When normalized with respect to appropriate material parameters, Φ and Ψ depend, save for a path history, only on functions $\sigma_e(\sigma_{ij})$ and $\dot{\varepsilon}_e(\dot{\varepsilon}_{ij})$, respectively, when made homogeneous of degree one with respect to their arguments. In particular, to reflect strain-hardening material behaviour, the potentials are allowed to depend on a scalar measure

$$\varepsilon_e = \int \dot{\varepsilon}_e \, \mathrm{d}t, \tag{3}$$

characterizing accumulated total strain which is in general path-dependent.

Explicitly with ε_e as a passive parameter in the Legendre transformation the potentials reduce to cf., e.g., Mroz (1973),

$$\Phi(\sigma_{ij}, \varepsilon_e) = \frac{\sigma_0}{n+1}\left(\frac{\sigma_e(\sigma_{ij})}{\sigma_0}\right)^{n+1} \varepsilon_e^{-n/m} \tag{4}$$

and

$$\Psi(\dot{\varepsilon}_{ij}, \varepsilon_e) = \sigma_0 \frac{n}{n+1}(\dot{\varepsilon}_e(\dot{\varepsilon}_{ij}))^{n+1/n}\varepsilon_e^{1/m} \tag{5}$$

where σ_e and $\dot{\varepsilon}_e$ are necessarily connected by

$$\sigma_e = \sigma_0 \dot{\varepsilon}_e^{1/n} \varepsilon_e^{1/m}. \tag{6}$$

As a consequence, eqn (1) may, in this form, be rewritten as

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_e \frac{\partial \sigma_e}{\partial \sigma_{ij}} \tag{7}$$

or

$$\sigma_{ij} = \sigma_e \frac{\partial \dot{\varepsilon}_e}{\partial \dot{\varepsilon}_{ij}}, \tag{8}$$

respectively.

Besides the material constant σ_0, the parameters m and n can be identified as exponents representing strain-hardening and rate-sensitivity or creep, respectively. This constitutive framework is thus suitable to represent primary (hardening) creep or rate-dependent plastic flow in general. In the limit when $m \to \infty$ nonlinear viscous flow or stationary creep is recovered while when $n \to \infty$ strain-hardening plastic flow is restored.

Although the shape of the flow function $\sigma_e(\sigma_{ij})$ has so far been left arbitrary, save for convexity, it has been implicitly assumed smooth together with its work-conjugate $\dot{\varepsilon}_e(\dot{\varepsilon}_{ij})$. Based on crystalline slip the existence of potential functions as here introduced has been discussed in depth by Rice (1970). For one thing in the time-independent plastic limit vertices are admissible and the present framework may then be included by introducing the notion of a subgradient cf., e.g., Maugin (1992), such that the strain rate is contained within a certain cone. Thus the analysis may be carried through also for nonassociated cases but for the stated simplicity only associated flow rules are considered further.

In order not to dim the visibility no effort is made to reproduce the most general homogeneous form of the function $\sigma_e(\sigma_{ij})$ for arbitrary anisotropic states. Instead, a straightforward quadratic function

$$\sigma_e = (\beta_{ijkl}\sigma_{ij}\sigma_{kl})^{1/2}, \tag{9}$$

is adopted and thus by (7)

$$\dot{\varepsilon}_{ij}/\dot{\varepsilon}_e = \beta_{ijkl}\sigma_{kl}/\sigma_e. \tag{10}$$

Then by the duality properties the inverse relation may be written as

$$\sigma_{ij}/\sigma_e = \alpha_{ijkl}\dot{\varepsilon}_{kl}/\dot{\varepsilon}_e \tag{11}$$

where α_{ijkl} is given by

$$\alpha_{ijkl}\beta_{klmn} = \tfrac{1}{2}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})$$

and the resulting work conjugate then by (7) and (11) as

$$\dot{\varepsilon}_e = (\alpha_{ijkl}\dot{\varepsilon}_{ij}\dot{\varepsilon}_{kl})^{1/2}. \tag{12}$$

It has been tacitly assumed above that the introduced potential Ψ is formally related to total strain-rate whatever the physical origin might be. At inelastic deformation of metals it is customary to assume that incompressibility prevails when elastic effects are insignificant and accordingly the dual potential Φ is insensitive to the mean stress $\sigma_{kk}/3$. The present framework then has to be slightly modified in the spirit of Hill (1987a) with eqn (7) retained but (8) replaced by

$$s_{ij} = \sigma_{ij} - \tfrac{1}{3}\sigma_{kk}\delta_{ij} = \sigma_e \frac{\partial \dot{\varepsilon}_e}{\partial \dot{\varepsilon}_{ij}}. \tag{13}$$

Then σ_e is a function of the stress deviator s_{ij}, though still being homogeneous of degree one, while $\dot{\varepsilon}_e$ is defined over the domain $\dot{\varepsilon}_{ij}$ so as to give $\partial\dot{\varepsilon}_e/\partial\dot{\varepsilon}_{kk} = 0$.

In case of both incompressibility and isotropy the introduced measures reduce to

$$\sigma_e = (\tfrac{3}{2} s_{ij} s_{ij})^{1/2}, \quad \dot{\varepsilon}_e = (\tfrac{2}{3} \dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij})^{1/2}, \tag{14}$$

which are of the familiar von Mises type and will be adopted in one of the applications below.

3. INDENTATION

3.1. *Formulation of the contact problem for a half-space*

The constitutive theory outlined above is general enough to analyse contact of a variety of solid materials which are of both structural or functional use. Some familiar applications which are of importance are shown in Fig. 1. With a self-similarity analysis in mind there

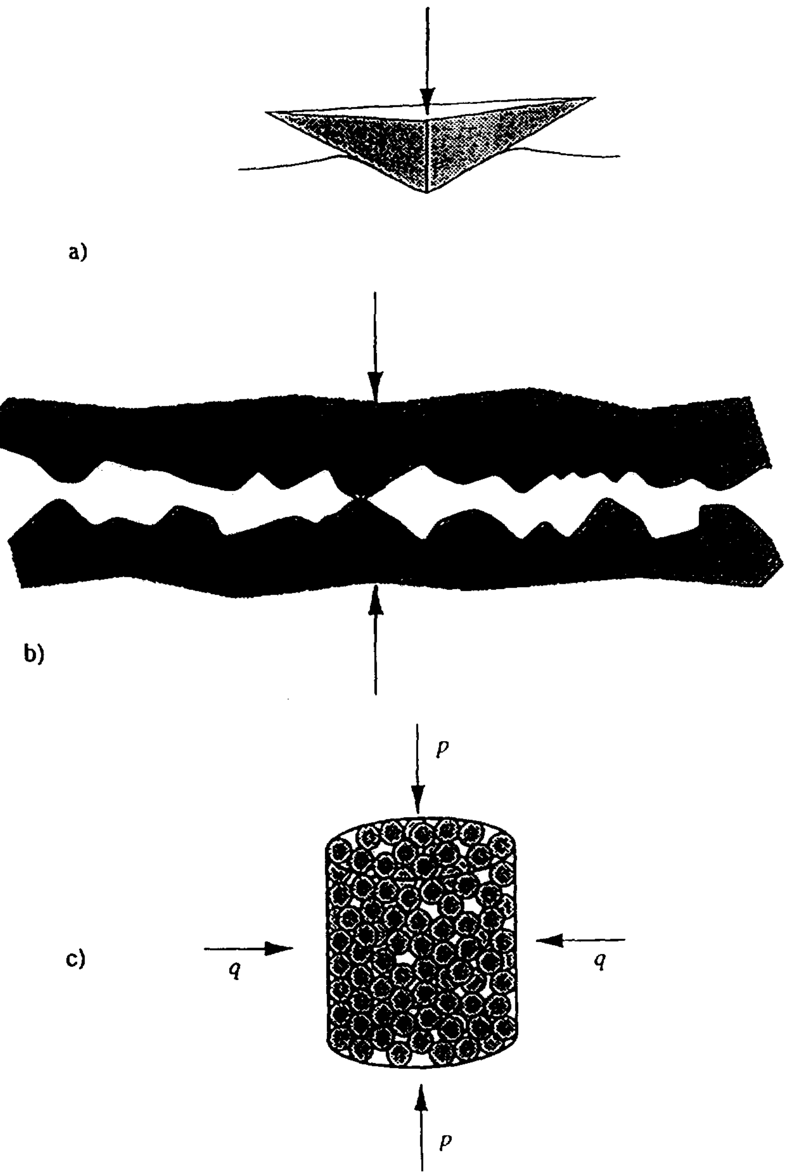

Fig. 1. Contact at (a) indentation (Berkovich), (b) flattening of asperities, (c) compaction.

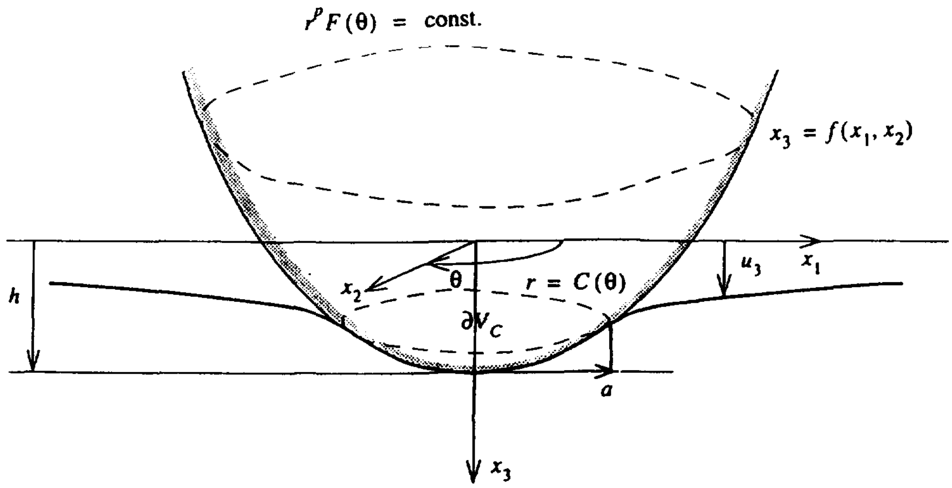

Fig. 2. Contact of a curved rigid indenter and a deformable halfspace.

is reason to dwell a little on the manner of formulating relevant boundary value problems. Small strain kinematics is to be used throughout the analysis as the contact region is assumed small compared to any characteristic length scale inherent in the present basic problem. It suffices then to consider two half-spaces with a local displacement boundary condition as in Hertz theory of elastic contact. Moreover, and also akin to Hertz theory, the problem may be made equivalent to that of a single half-space of combined material properties impressed by a rigid punch as will be displayed in detail below. To gain first insight and clarity though the fundamental problem of contact between a curved rigid indentor and a flat deformable solid is analysed first.

The profile of the indenter is represented by a relation $x_3 = f(x_1, x_2)$ as depicted in Fig. 2 where at the origin $f(0,0) = 0$. Further the profile function f is assumed convex and positively homogeneous of degree p, i.e.,

$$f(\alpha x_1, \alpha x_2) = \alpha^p f(x_1, x_2). \tag{15}$$

In some situations it proves convenient to use cylindrical polars, i.e.,

$$f = F(\theta)D(r/D)^p, \tag{16}$$

where D is a curvature parameter and the die angular function $F(\theta)$ may be normalized as $F(0) = 1$ without loss of generality.

In the employed form of headshapes it may be noted in passing that several familiar cases of importance are recognized. Thus, in the axisymmetric case when $F(\theta) \equiv 1$ the case $p = 1$ corresponds to a cone and $p = 2$ to a sphere of curvature $2/D$. In nonsymmetric cases such as pyramidal indentation with, e.g., a square cross section (Vickers), the shape is defined by $p = 1$ and

$$F(\theta) = \cos\theta, \quad 0 \leqslant \theta \leqslant \pi/4 \tag{17}$$

for a necessary and sufficient interval of definition with regard to symmetry. Likewise for a triangular cross section (Berkovich), Fig. 1a, the corresponding symmetry interval is $0 \leqslant \theta \leqslant \pi/3$.

Returning now to the boundary value problem it is assumed that the indenter is rigid and pressed normally onto a half-space, occupying $x_3 \geqslant 0$, according to Fig. 2. Then, by the imposed displacement h, the local boundary conditions may be formulated as

$$u_3 = h - f(x_1, x_2), \quad \sigma_{13} = \sigma_{23} = 0 \quad \text{at} \quad x_3 = 0, \quad (x_1, x_2) \in \partial V_c, \tag{18}$$

where ∂V_c is the so far unknown contact region which must be determined as part of the solution. Explicitly, if the contour is expressed as $r = C(\theta)$, the evolution of $C(\theta)$, as a function of h, Fig. 2, is to be determined.

It should be emphasized that, by the homogeneous boundary conditions in (18), frictionless indentation is prescribed. The significance of this assumption in practice, of course, varies with the circumstances. It is adopted here for the single purpose to later demonstrate explicit solutions. In the present self-similarity framework, however, there are no formal difficulties to accommodate cases of adhesion, Spence (1968), and Coulomb friction, Borodich (1993), as studied by these writers for linear and nonlinear elasticity, respectively. Further it has so far been tacitly assumed that linear kinematics applies. In particular, in the case of sharp indenters, however, large deformations may be appreciable and should deserve attention especially as the homogeneity properties prescribed above may be relaxed for standard profiles. Some details of these issues have been discussed by Bower *et al.* (1993) for power-law creep but more general cases of nonlinear material behaviour and indenter profiles will be left for further exploration.

Posing the boundary value problem as of rate type, the field equations together with the boundary conditions may now be summarized as

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial \dot{u}_i}{\partial x_j} + \frac{\partial \dot{u}_j}{\partial x_i}\right), \tag{19}$$

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0, \tag{20}$$

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_e \frac{\partial}{\partial \sigma_{ij}} \sigma_e(\sigma_{kl}) \quad \text{or} \quad \sigma_{ij} = \sigma_e \frac{\partial}{\partial \dot{\varepsilon}_{ij}} \dot{\varepsilon}_e(\dot{\varepsilon}_{kl}), \quad \sigma_e = \sigma_0 \dot{\varepsilon}_e^{1/n} \varepsilon_e^{1/m}, \tag{21}$$

$$\dot{u}_3 = \dot{h}, \quad \sigma_{13} = \sigma_{23} = 0, \quad r \leqslant C(\theta) \tag{22}$$

$$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0 \qquad r > C(\theta) \tag{23}$$

where $\dot{\varepsilon}_e$ and ε_e are defined by eqns (12) and (3), respectively. Equations (19), (20) and (21) correspond to compatibility, equilibrium and constitutive law, in this order and in obvious notation. The boundary conditions (23) correspond to the free surface and the remaining remote conditions have to be considered separately in every individual situation.

3.2. *Self-similarity and the reduced problem*

Essentially when predicting self-similarity to apply in the present setting the resulting contact contour must be invariant with respect to natural time and expand in a spatially self-similar way. With this prerequisite the contact contour may be expressed as

$$r = a\tilde{C}(\theta), \quad \tilde{C}(0) = 1, \tag{24}$$

where, as shown in Fig. 2, a represents a reference contact radius in the direction $\theta = 0$. At axisymmetry $\tilde{C}(\theta) \equiv 1$ and the unknown contact region may be simply determined by the scalar value a but, in general, $\tilde{C}(\theta)$ must be found independently and the validity of the decomposition in (24) proven *a posteriori*.

To proceed in the spirit of the axisymmetric analyses by Storåkers and Larsson (1994) and Biwa and Storåkers (1995) pure kinematics is considered first with suitable *a priori* scalings

$$x_i = a\tilde{x}_i, \tag{25}$$

$$\dot{u}_i(x_k, a) = \dot{h}\tilde{u}_i(\tilde{x}_k) \tag{26}$$

and

$$\dot{\varepsilon}_{ij}(x_k, a) = (\dot{h}/a)\tilde{\varepsilon}_{ij}(\tilde{x}_k). \tag{27}$$

Thus the scaled variables are assumed to be independent of the representative radius a which again has to be verified *a posteriori*.

With the scaled velocity field $\tilde{u}_i(\tilde{x}_k)$ thus introduced, the inhomogeneous rate boundary condition (22) now reduces to

$$\tilde{u}_3 = 1, \quad \tilde{x}_3 = 0, \quad \tilde{r} \leqslant \tilde{C}(\theta) \tag{28}$$

which then formally corresponds to flat die indentation.

The resulting velocity field must, however, be integrated to satisfy the original displacement boundary condition (18), viz.

$$\int_0^t \tilde{u}_3 \dot{h}\, \mathrm{d}t = h - f(x_1, x_2). \tag{29}$$

Using polar coordinates and a variable transformation from t to a then by (24)

$$h(r/\tilde{C}(\theta)) - \int_0^{r/\tilde{C}(\theta)} \tilde{u}_3(r/s)h'(s)\,\mathrm{d}s = F(\theta)r^p/D^{p-1}, \quad h'(a) = \frac{\mathrm{d}h}{\mathrm{d}a} \tag{30}$$

for any fixed θ, with the current contact contour defined by (24).

Regarded as an integral equation for $h = h(a)$ the solution to (30) is readily obtained as

$$h(a) = \frac{F(\theta)}{c^p(\theta)} \frac{a^p}{D^{p-1}} \tag{31}$$

with the eigenfunction $c^p(\theta)$ given by

$$c^p(\theta) = \left(\frac{1}{\tilde{C}(\theta)}\right)^p - p\int_{\tilde{C}(\theta)}^{\infty} \frac{\tilde{u}_3}{\tilde{r}^{p+1}}\,\mathrm{d}\tilde{r}. \tag{32}$$

To ensure a unique relation of the depth h for the contour by (31) it is required that

$$c^p(\theta) = F(\theta)c^p(0). \tag{33}$$

As a consequence by (16), (18) and (31) the displacements under the die may be written as

$$u_3 = h(1 - c^p(\theta)(r/a)^p) \tag{34}$$

and in particular at the contour "piling-up" or "sinking in" occurs whether $c^p(\theta)(\tilde{C}(\theta))^p$ exceeds unity or not by (24).

As only kinematics has been considered so far it remains to fully formulate the resulting intermediate flat die problem and also to actually solve it. To this end further scaling of stresses and strains may be introduced as

$$\sigma_{ij}(x_k, a) = \sigma_0(\dot{h}/a)^{1/n}(h/a)^{1/m}\tilde{\sigma}_{ij}(\tilde{x}_k) \tag{35}$$

and

$$\varepsilon_{ij}(x_k, a) = (h/a)\hat{\varepsilon}_{ij}(\tilde{x}_k). \tag{36}$$

Then the field equations reduce to

$$\tilde{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} + \frac{\partial \tilde{u}_j}{\partial \tilde{x}_i}\right), \tag{37}$$

$$\frac{\partial \tilde{\sigma}_{ij}}{\partial \tilde{x}_j} = 0 \tag{38}$$

and

$$\tilde{\varepsilon}_{ij} = \tilde{\varepsilon}_e \frac{\partial \tilde{\sigma}_e}{\partial \tilde{\sigma}_{ij}}, \quad \tilde{\sigma}_{ij} = \tilde{\sigma}_e \frac{\partial \tilde{\varepsilon}_e}{\partial \tilde{\varepsilon}_{ij}}, \quad \tilde{\sigma}_e = \hat{\varepsilon}_e^{1/m}\tilde{\varepsilon}_e^{1/n}, \tag{39}$$

from (19), (20), and (21), respectively, and the non-trivial boundary condition is given by (28).

In particular by integration of (27) and a variable transformation by (31) the scaling introduced in (36) may be rewritten as

$$\hat{\varepsilon}_e = p\tilde{\rho}^{p-1}\int_{\tilde{\rho}}^{\infty} \frac{\tilde{\varepsilon}_e}{\tilde{\rho}_t^p}\,\mathrm{d}\tilde{\rho}_t. \tag{40}$$

This transformation procedure, as introduced by Biwa and Storåkers (1995) for the case of plastic flow theory, is of vital interest as integration in (40) may be carried out along radial rays extending from points $\tilde{x}_i$, where $\tilde{\rho}^2 = \tilde{x}_i\tilde{x}_i$, to infinity. In essence then the self-similarity formulation at hand has replaced time and material history dependence by spatially nonlocal dependence and transformed it to an intermediate flat die problem which in turn will generate similarity solutions to the original indentation problem.

Thus by the scaling of rates of displacements and strains and straightforward cumulative superposition using again the variable transformation by (31) there results

$$u_i(x_k, a) = ph\left(\frac{\rho}{a}\right)^p \int_{\rho/a}^{\infty} \tilde{u}_i(\tilde{\rho})\tilde{\rho}^{-(p+1)}\,\mathrm{d}\tilde{\rho} \tag{41}$$

and

$$\varepsilon_{ij}(x_k, a) = p\frac{h}{a}\left(\frac{\rho}{a}\right)^{p-1} \int_{\rho/a}^{\infty} \tilde{\varepsilon}_{ij}(\tilde{\rho})\tilde{\rho}^{-p}\,\mathrm{d}\tilde{\rho}, \tag{42}$$

respectively, where the required integrations are to be carried along radial rays in analogy with (40).

At interpretation of indentation tests the main interest concerns the dependence of the resulting mean pressure on the indentation depth or the contact region. In the present formulation then the scaling introduced by (35) first generates the total load L as

$$L = a^2\sigma_0(\dot{h}/a)^{1/n}(h/a)^{1/m}\int(-\tilde{\sigma}_{33})\,\mathrm{d}\tilde{A} \tag{43}$$

where the total contact area, A, is scaled as

$$A = a^2\tilde{A} = a^2 \int_0^{2\pi} \frac{[\tilde{C}(\theta)]^2}{2} \mathrm{d}\theta \tag{44}$$

remembering (24).

As a principal finding by (31), the indentation depth and the representative radius a are separable and using also (33) the load may be expressed solely by aid of a as

$$L = a^2\sigma_0 \left[\frac{1}{c^p(0)}\left(\frac{a}{D}\right)^{p-1}\right]^{1/m} \left[\frac{1}{c^p(0)}\frac{p\dot{a}}{D}\left(\frac{a}{D}\right)^{p-2}\right]^{1/n} \int(-\tilde{\sigma}_{33})\,\mathrm{d}\tilde{A}. \tag{45}$$

Thus, the mean pressure (hardness) at indentation may be summarized, symbolically, as

$$\frac{L}{A} = \sigma_0\alpha(m,n,p)\left[\beta_m(m,n,p)\left(\frac{a}{D}\right)^{p-1}\right]^{1/m}\left[\beta_n(m,n,p)\frac{\dot{a}}{D}\left(\frac{a}{D}\right)^{p-2}\right]^{1/n} \tag{46}$$

where the combined parameters α, β_m and β_n are, in general, functions only of the material exponents m and n and of the indentation geometry represented by p and a weighted value of $F(\theta)$, though not on the magnitude of indentation. Instead, the separable dependence on the contact radius and its rate is quite clear from (46).

Thus it was described in some detail above how in the original indentation problem involving a moving boundary space and time dependence could be transformed into a stationary problem and where an indenter of arbitrary profile could be replaced by that of a flat die. Accordingly there are evidently advantages connected with this approach, although technical entanglements might unfold when explicit solutions are aimed at in particular cases. First in a general situation the shape of the contact contour is unknown and when it comes to inelastic material behaviour and in particular, when hardening is at issue, the reduced constitutive equation contains a nonlocal ingredient. The difficulties are by no means unsurmountable though as will be explicitly demonstrated below.

3.3. *Remarks on nonlinear elasticity and deformation theory of plasticity*

It was natural within the purely viscoplastic framework to apply a rate formulation leading to the intermediate flat die problem to solve. In case of nonlinear elasticity or, alternatively, deformation theory of plasticity, this theory is conventionally based on total strain or deformation. The possibility to apply self-similarity at indentation has been drawn upon also for such materials in case of power-law behaviour and small strains and was applied by Hill *et al.* (1989) to fully solve the Brinell problem in particular with the view of deformation theory of plasticity in mind. Subsequently indentation by a more general class of profiles was discussed by Borodich (1989) and Storåkers (1989), though still within nonlinear elasticity.

Although a direct formulation based on total displacements was successfully applied by Hill *et al.* (1989), there are reasons at least from a computational point of view to discuss a competing rate formulation also at nonlinear elasticity. To this end the constitutive equation then reads originally

$$\varepsilon_{ij} = \varepsilon_e \frac{\partial\sigma_e}{\partial\sigma_{ij}} \tag{47}$$

and

$$\sigma_{ij} = \sigma_e \frac{\partial \varepsilon_e}{\partial \varepsilon_{ij}} \tag{48}$$

by simply replacing strain rates with strains in analogy with (7) and (8).

The rate-type counterparts are

$$\dot{\varepsilon}_{ij} = M_{ijkl}\dot{\sigma}_{kl}, \quad M_{ijkl}(\sigma_{pq}) = \varepsilon_e \frac{\partial^2 \sigma_e}{\partial \sigma_{ij}\, \partial \sigma_{kl}} + \frac{1}{\mathrm{d}\sigma_e/\mathrm{d}\varepsilon_e} \frac{\partial \sigma_e}{\partial \sigma_{ij}} \frac{\partial \sigma_e}{\partial \sigma_{kl}} \tag{49}$$

or, inversely,

$$\dot{\sigma}_{ij} = L_{ijkl}\dot{\varepsilon}_{kl}, \quad L_{ijkl}(\varepsilon_e) = \sigma_e \frac{\partial^2 \varepsilon_e}{\partial \varepsilon_{ij}\, \partial \varepsilon_{kl}} + \frac{\mathrm{d}\sigma_e}{\mathrm{d}\varepsilon_e} \frac{\partial \varepsilon_e}{\partial \varepsilon_{ij}} \frac{\partial \varepsilon_e}{\partial \varepsilon_{kl}} \tag{50}$$

where $\sigma_e = \sigma_0 \varepsilon_e^{1/m}$.

With this as a background it is now clear by analogy that a full solution may be obtained by first solving the flat die problem with stress rates as primary variables and subsequently apply cumulative superposition. Some qualitative conclusions may then be drawn.

Save for linear elasticity which arises as a special case the tangent moduli in (49) or (50) in general depend on total stress or strain. Thus with the reduced problem as a start it will contain a nonlocal ingredient as in the case of strain-hardening plasticity above. In contrast, however, in a total displacement formulation as used by Hill *et al.* (1989), the problem must be solved for the displacements prescribed on the boundary for every specific case as represented by the index p, while the corresponding flat die problem here may be solved once and for all with p only constituting a parameter. Perhaps the main virtue of the present rate formulation is, however, that the contact contour, through the eigenvalue c^p, eqn (32), generating the invariant indentation depth, eqn (31), may be directly determined by cumulative superposition implying integration, while in a total displacement formulation a trial and error procedure to determine contour values has to be resorted to. The latter was the case in the plastic deformation theory analysis by Hill *et al.* (1989) and would persist also in an attempt to explicitly solve the corresponding creep problem posed by Hill (1992) in the same spirit.

4. ILLUSTRATIVE APPLICATIONS

4.1. *Ellipsoidal indentation of Newtonian fluids*

In order to gain explicit insight in the procedure proposed it appears suitable to first discuss a problem for which a nontrivial analytical solution may be expected. Such are, with no known exceptions, confined to linear elastic or viscous solids with simple contact contours like circles and strips but a genuinely three-dimensional case pertains to ellipsoidal indentation. The latter problem has since long been analysed and resolved for the linear elastic case, cf., e.g., Johnson (1985) or Hills *et al.* (1993) and is here reexamined to apply to a linearly viscous half-space and in particular to illustrate the background and delineate the formulae (32), (33) and (46) above.

The material law in question now reads

$$s_{ij} = 2\mu\dot{\varepsilon}_{ij} \tag{51}$$

where μ is the shear viscosity. This class of material behaviour is obtained as a special type of that considered presently, with $m \to \infty$, $n = 1$ and $\sigma_0 = 3\mu$ in (21).

The indenter profile is assumed to be locally ellipsoidal, and as in (15) with $p = 2$,

$$f = Ax_1^2 + Bx_2^2, \quad A \leqslant B \tag{52}$$

or alternatively by (16)

$$f = F(\theta)r^2/D \tag{53}$$

where

$$F(\theta) = \cos^2\theta + (B/A)\sin^2\theta, \quad A = 1/D, \tag{54}$$

with notation as in Fig. 2.

With the strategy outlined above, the problem at hand is first reduced to flat indentation of an incompressible linear elastic half-space though yet with an unknown contact contour, $\tilde{C}(\theta)$ to be determined. To proceed, in general, an initial conjecture is needed at the outset and to be refined to satisfy the requirement (33). Here, it is reasonable to assume that the contour is given by an ellipse, with the unit major axis in the x_1-direction and the minor one $1/\sqrt{1-e^2}$ in the x_2-direction, where e is the eccentricity of the ellipse to be determined.

The solution to the stated reduced elastic Boussinesq flat die problem is known since the time of Hertz and the reduced vertical displacement on the free surface, $\tilde{u}_3 = u_3/h$, reads, cf. Johnson (1985, p. 64),

$$\tilde{u}_3(\tilde{x}_1, \tilde{x}_2, 0) = \frac{1}{2K(e)}\int_{\lambda_1}^{\infty} \frac{\mathrm{d}w}{\{(1+w)(1+w-e^2)w\}^{1/2}}, \tag{55}$$

where $\tilde{x}_\alpha = x_\alpha/a$ and $K(e)$ is the complete elliptic integral of the first kind with the modulus e and λ_1 is the positive solution of

$$\frac{\tilde{x}_1^2}{1+\lambda} + \frac{\tilde{x}_2^2}{1-e^2+\lambda} = 1. \tag{56}$$

Equations (55) and (56) are then to be substituted into the formula (32) and the condition (33) imposed. In doing so (32) is first rewritten by partial integration as

$$c^p(\theta) = -\int_{\tilde{C}(\theta)}^{\infty} \frac{\mathrm{d}\tilde{u}_3}{\mathrm{d}\tilde{r}} \frac{1}{\tilde{r}^p} \mathrm{d}\tilde{r}. \tag{57}$$

Then when $p = 2$ and with $\tilde{C}(\theta) = \cos^2\theta + (1-e^2)\sin^2\theta$ implied by the assumption made, substitution of (55) and (56) into (57) readily yields

$$c^2(\theta) = \frac{\cos^2\theta}{e^2K(e)}\{K(e)-E(e)\} + \frac{\sin^2\theta}{e^2K(e)}\left\{\frac{E(e)}{1-e^2} - K(e)\right\}, \tag{58}$$

where $E(e)$ is the complete elliptic integral of the second kind.

In order for (58) to satisfy (33), it is required for any θ that

$$\cos^2\theta + \frac{A}{B}\sin^2\theta \equiv \cos^2\theta + \frac{E(e)/(1-e^2)-K(e)}{K(e)-E(e)}\sin^2\theta, \tag{59}$$

and, accordingly, when given the indenter parameters A and B in the original problem, the unknown contour parameter (eccentricity of the contact ellipse) e must be chosen as

$$\frac{B}{A} = \frac{E(e)/(1-e^2)-K(e)}{K(e)-E(e)} \tag{60}$$

to satisfy (59).

Thus, it is seen that the sought-for contour is indeed an ellipse, which shape complies with the above formula. Moreover, for the indentation depth h and the contact size a, the relation from (31) follows as

$$h(a) = \frac{e^2 K(e)}{K(e)-E(e)} \frac{a^2}{D}, \tag{61}$$

with e given by (60).

The stress field may now be directly obtained from the flat die solution and other variables such as total displacements for the original curved die problem follow as well when supplemented by cumulative superposition as in (41) and (42) although details are suppressed here.

The pressure distribution in the contact region is found to be, drawing upon the linear elastic solution,

$$\tilde{p}(\tilde{x}_1, \tilde{x}_2) = \tilde{p}_0 \frac{1}{\sqrt{1-\tilde{x}_1^2-\tilde{x}_2^2/(1-e^2)}} \tag{62}$$

in the scaled form, where the central value is given by

$$\tilde{p}_0 = \frac{2}{3} \frac{1}{K(e)\sqrt{1-e^2}}. \tag{63}$$

Likewise, the reduced load is

$$\tilde{L} = \int \tilde{p}\, \mathrm{d}\tilde{A} = \frac{4}{3} \frac{\pi}{K(e)}. \tag{64}$$

Then, with the introduced scaling, the total load for the original problem is

$$L = a^2 \sigma_0 \left(\frac{\dot{h}}{a}\right) \tilde{L} = \frac{4}{3} \pi a^2 \sigma_0 \left(\frac{\dot{h}}{a}\right) \Big/ K(e), \tag{65}$$

or by using (61) and with $\sigma_0 = 3\mu$,

$$L = \frac{8\pi e^2}{K(e)-E(e)} \mu \frac{a^2 \dot{a}}{D}. \tag{66}$$

Thus the relation between the load and the impression magnitude or the contact area depends on the factor e, which in turn is determined by the shape of the ellipsoidal punch according to (60). In the limit when the punch becomes locally spherical, i.e., as $e \to 0$, by the use of formulae

$$\lim_{e\to 0} K(e) = \lim_{e\to 0} E(e) = \pi/2, \tag{67}$$

$$\lim_{e\to 0} (K(e)-E(e))/e^2 = \lim_{e\to 0} (E(e)-(1-e^2)K(e))/e^2 = \pi/4, \tag{68}$$

the obtained results reduce to

$$h(a) = 2a^2/D \tag{69}$$

and

$$L = 32\mu a^2 \dot{a}/D. \tag{70}$$

For the degenerate case of spherical indentation the results are in conformity with well known expressions from viscoelastic contact theory, in the limiting case of vanishing elasticity, as reviewed by Johnson (1985, p. 192).

Turning back now to a general case of an ellipsoidal punch, indentation of the viscous half-space by a prescribed constant load, say, implies growth of the contact region given by (66), i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{a^3}{D}\right) = \frac{K(e)-E(e)}{e^2}\frac{3L}{8\pi\mu}. \tag{71}$$

From a virgin state, $a(0) = 0$, (71) may be integrated to yield

$$a^3(t)/D = \frac{K(e)-E(e)}{e^2}\frac{3L}{8\pi\mu}t, \tag{72}$$

i.e., the contact radius grows in scale as $t^{1/3}$ under a constant load.

4.2. *Spherical indentation of viscoplastic solids*

As has already been emphasized, spherical indentation is of interest in many applications. For one thing the classical Brinell test, although being of metallurgical origin, has attracted many investigators from a mechanics point of view. More recently, the main features have been analysed in detail for deformation theory of plasticity by Hill *et al.* (1989) and in the present spirit by Biwa and Storåkers (1995) for plastic flow theory and by Storåkers and Larsson (1994) for stationary creep. In the present viscoplastic setting the two limiting cases then reduce to rate independent strain-hardening solids and power law viscous solids respectively.

As to the degree of increasing complexity the special case of power law stationary creep, i.e., $m \to \infty$ eqn (6), was solved by Storåkers and Larsson (1994) for a spherical indenter, the Brinell problem, as illustrated in Fig. 3a. The corresponding reduced problem then relates to a nonlinear elastic solid indented by a flat circular punch, Fig. 3b, which is a version of what in the creep context is known as Hoff's analogy (Hoff 1954). A finite element procedure was devised and the problem solved for a range of power law exponents

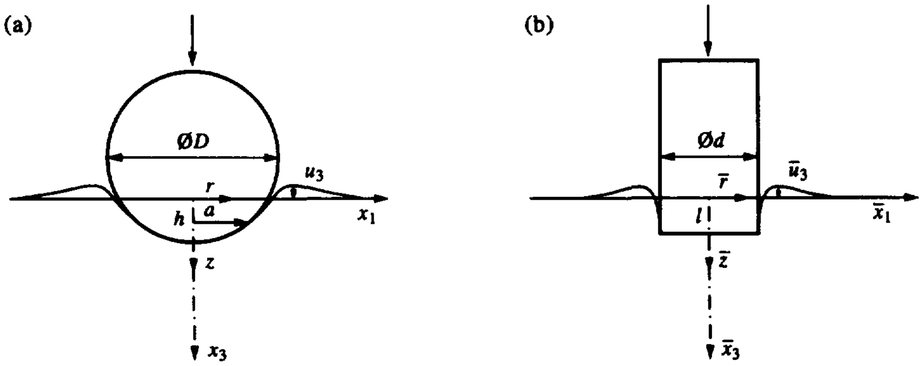

Fig. 3. Spherical Brinell indentation and the related flat die Boussinesq problem.

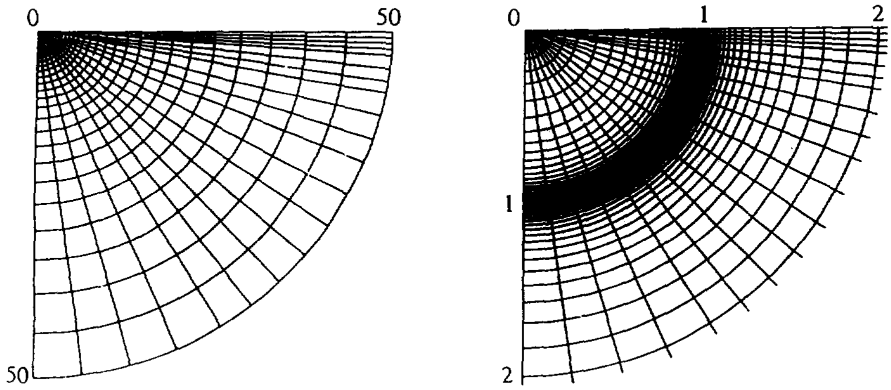

Fig. 4. Complete finite element mesh, $\rho/a \leqslant 50$, designed for the reduced flat die problem with details in the region $\rho/a \leqslant 2$.

and, in particular, for $n \to \infty$ when perfect plasticity applies. Although the solution involves a singularity at the contact boundary similar to an HRR crack tip field (Hutchinson 1968, Rice and Rosengren 1968), no fundamental difficulties were encountered but instead it proved advantageous in particular to determine the contact boundary, i.e., the eigenvalue c^2, by a cumulative procedure as in eqns (31), (32) above with $p = 2$ and $\tilde{C}(\theta)$ and $F(\theta)$ set to unity. The Brinell problem was subsequently attacked by Biwa and Storåkers (1995) but now by plastic flow theory, i.e., $n \to \infty$ eqn (6), as a basis. Then as strain-hardening is present a nonlocal as well as a nonlinear constitutive law needs to be coped with in the reduced problem in particular with eqns (39) and (40) above in mind. This difficulty was, however, readily resolved by a finite element procedure with 13,882 degrees of freedom where an iterative procedure was used to determine the accumulated strain by integration along radial rays. The finite element mesh originally designed by Biwa and Storåkers (1995) is shown in Fig. 4 and the associated computational strategy was followed closely here involving no further fundamental difficulties.

When analysing the general case here for a sphere, $p = 2$, the reduced constitutive equations (39), (40) read

$$\tilde{s}_{ij} = \frac{2}{3}\hat{\varepsilon}_e^{1/m}\tilde{\varepsilon}_e^{(1/n)-1}\tilde{\varepsilon}_{ij}, \quad \hat{\varepsilon}_e = 2\tilde{\rho}\int_{\tilde{\rho}}^{\infty}\frac{\tilde{\varepsilon}_e}{\tilde{\rho}_t^2}\,d\tilde{\rho}_t \tag{73}$$

where, accordingly, incompressibility has been assumed and the Levi–Mises flow rule adopted. With the earlier computational procedures as described above by Storåkers and Larsson (1994) for creep and, in particular, for plastic flow, with hereditary behaviour present as in $(73)_2$, by Biwa and Storåkers (1995), to investigate the fully viscoplastic case is a fairly straight-forward procedure. It was found by both Storåkers and Larsson (1994) and Biwa and Storåkers (1995) that, at large values of n and m, respectively, deformation will localize, which might cause convergence problems. An immediate remedy was, however, to reduce the finite element mesh in Fig. 4. In this case the complete mesh, $\tilde{\rho} = 50$, was used for $1/m + 1/n \geqslant 0.25$ while for $0.1 \leqslant 1/m + 1/n \leqslant 0.25$ the remote boundary was reduced to $\tilde{\rho} = 10$. No significant differences of results appeared at the mesh reduction. Convergence proved to be especially sensitive to the power law exponent m and it proved efficient for higher m-values in the iteration procedure to use a parameter tracking strategy starting at $m = 1$.

It is not the present intention to carry out a detailed parameter study but instead some representative results and formulae, believed to be of practical concern, will be displayed based on the finite element results. Thus the surface shape of a deformed half-space is a true characteristic of the hardening properties. With the eigenvalue $c^2(m, n)$ in mind the profile under a spherical indenter may be written according to eqn (34) as

$$u_3 = h[1-c^2(r/a)^2], \quad r \leqslant a \tag{74}$$

and it is obvious that the resulting transverse displacement at the contact contour is above the original surface when $c^2 > 1$, piling-up, and below, sinking-in, otherwise. It has long been known for a variety of experimental observations, cf., e.g., Norbury and Samuel (1928), that the residual plastic impression remains below the surface for annealed metals and alloys at approximately $m = 3$ while at various degrees of cold work piling-up occurs.

The invariant $c^2(m,n)$ is thus of primary importance remembering both the fundamental kinematic result (31) and the elucidated indentation depth eqn (74). It has been determined here based on the viscoplastic solution of the reduced stationary problem supplemented by cumulative superposition and results are shown in Fig. 5 as function of the power-law exponents m and n. Earlier findings for the purely plastic strain-hardening case, $n \to \infty$, by Biwa and Storåkers (1995) for flow theory and by Hill *et al.* (1989) for deformation theory are also depicted together with the pure creep case, $m \to \infty$, by Storåkers and Larsson (1994).

In their analysis based on plastic flow theory, it was observed by Biwa and Storåkers (1995) that to a very close agreement $c^2(m) = c^2(n)$ as compared to the creep results by Storåkers and Larsson (1994). As a consequence the presently determined combined cases are here plotted as a function of $1/m+1/n$ and it is rather striking that within a very good accuracy $c^2 = c^2(1/m+1/n)$. Some further background to this pragmatic finding may be found from eqn (73). Thus if $\hat{\varepsilon}_e = \tilde{\varepsilon}_e$, then the field equations are fully satisfied as a single function of $1/m+1/n$ which for one thing generates the resulting c^2-value. For this to be exact, however, partial proportional effective straining, viz. $\dot{\varepsilon}_e = (\dot{h}/a)g(h/a)\varepsilon_e$, g being an

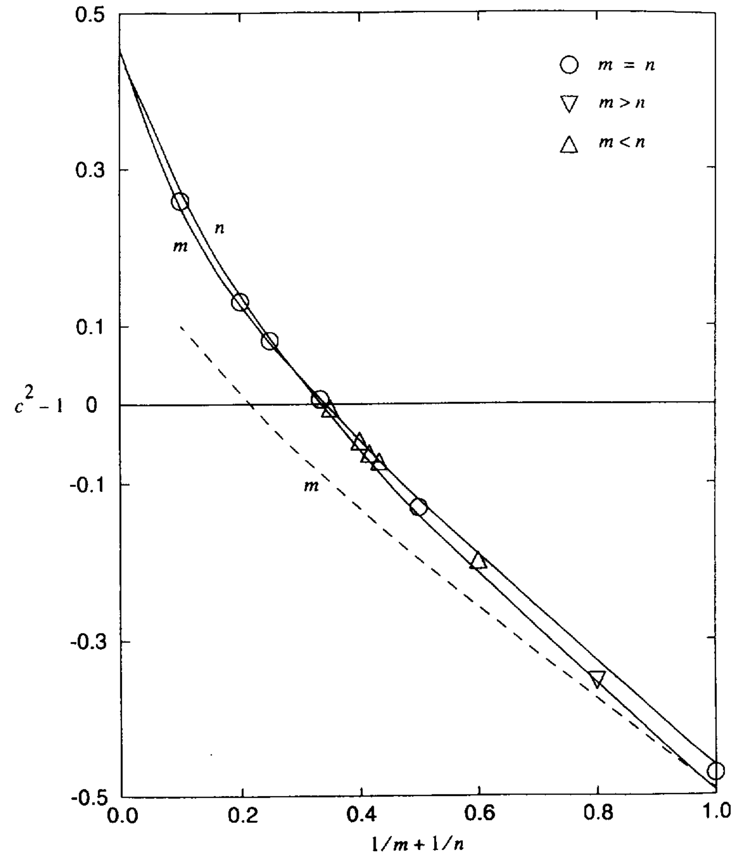

Fig. 5. Relative height $c^2-1 = -u_3/h$ above the contour as function of $1/m+1/n$ at spherical indentation of viscoplastic materials. General viscoplasticity, present results, (○) $m = n$, (▽) $m > n$, (△) $m < n$, (— m) plastic flow theory, Biwa and Storåkers (1995), (— n) creep theory, Storåkers and Larsson (1994), (- - -m) deformation theory of plasticity, Hill *et al.* (1989).

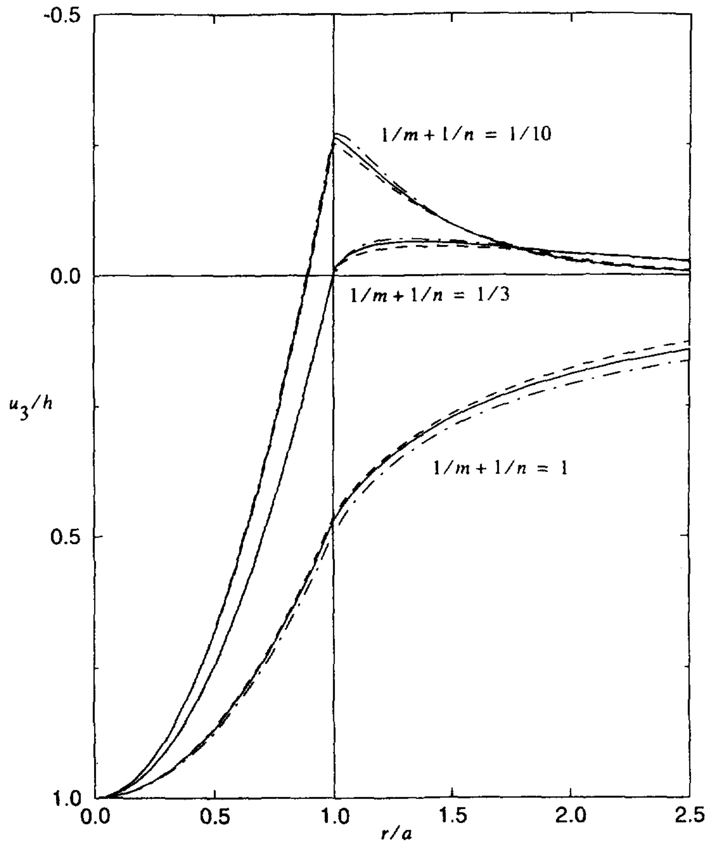

Fig. 6. Deformed surface shape at spherical indentation of viscoplastic materials for $1/m+1/n = 1$, 1/3, 1/10. (—) general viscoplasticity ($m = n$) present results, (- - -) plastic flow theory ($n \to \infty$) Biwa and Storåkers (1995), (- · -) creep theory ($m \to \infty$) Storåkers and Larsson (1994).

arbitrary function, is required as may be readily seen from $(73)_2$ remembering (27) and (40).

From the set of values shown in Fig. 5, it may be observed that at $1/m+1/n \approx 1/3$ there is a transition from sinking-in to piling-up. Similar results have also been determined based on the asymptotic procedure by Ogbonna *et al.* (1995). The agreement is good in general although all their c^2-values fall below the present ones by about 5%.

As to field results it has been shown earlier by Biwa and Storåkers (1995) for strain-hardening plasticity that effective strain trajectories are indeed similar to the corresponding ones for creep, Storåkers and Larsson (1994). Here some representative deformed profiles are shown in Fig. 6. Thus it may be observed that again the results essentially depend only on $1/m+1/n$. In particular for high values of the exponents pronounced piling-up occurs at the contact edge. The very refined finite element mesh in the vicinity of the contact boundary as shown in Fig. 4, was in fact designed in particular to resolve the almost singular behaviour exhibited.

The mean pressure as given by eqn (46) for a spherical die, $p = 2$, reduces to

$$\frac{L}{\pi a^2} = \sigma_0 \alpha(m,n)\left[\beta_m(m,n)\left(\frac{a}{D}\right)\right]^{1/m}\left[\beta_n(m,n)\frac{\dot{a}}{D}\right]^{1/n} \tag{75}$$

where again the combination of parameters α, β_m, β_n is to be determined from the flat die solution followed by cumulative superposition.

Earlier results have given for m, n individually, Biwa and Storåkers (1995) and Storåkers and Larsson (1994), respectively. Obviously for $m, n \to \infty$ the values for the reduced

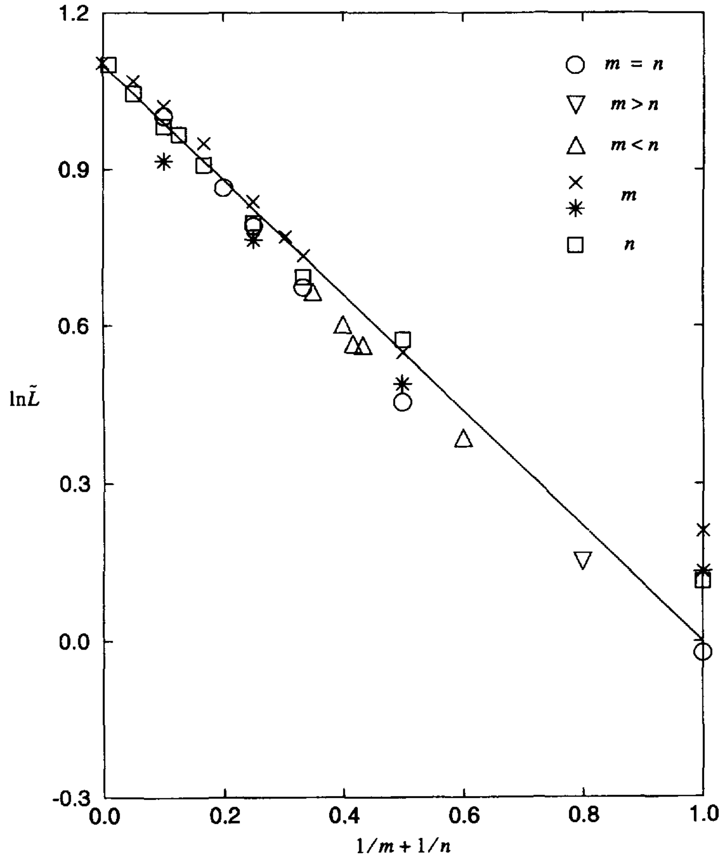

Fig. 7. Reduced mean pressure $\tilde{L}$ as function of $1/m+1/n$ at spherical indentation of viscoplastic materials. General viscoplasticity, present results, (○) $m = n$, (▽) $m > n$, (△) $m < n$, (×) plastic flow theory $n \to \infty$, Biwa and Storåkers (1995), (□) creep theory $m \to \infty$, Storåkers and Larsson (1994), (*) deformation theory of plasticity $n \to \infty$, Hill *et al.* (1989), (—) present fit for general viscoplasticity (eqns (76, 78)).

load coincide as for perfect plasticity while for m, $n = 1$ a factor 3 is approximately at variance and exact only for linear elasticity in contrast to homogenity of degree one.

Thus it proved useful here to introduce a factor $(n+2)/n$ in analogy with (45) to scale the results as

$$\frac{L}{\pi a^2} = \frac{n+2}{n}\sigma_0 \left(\frac{a}{D}\right)^{1/m} \left(\frac{\dot{a}}{D}\right)^{1/n} \tilde{L} \tag{76}$$

based on the reduced field equations to determine $\tilde{L}(m, n)$.

The corresponding outcome is shown in Fig. 7 where the reduced mean pressure is again shown as a function of $1/m+1/n$. By introduction of the factor $(n+2)/n$ in (76) it may be seen that the reduced load is also here virtually governed by the combined power law exponents. The corresponding results for deformation theory of plasticity by Hill *et al.* (1989) are also shown for comparison. Presently if choosing, in eqn (45),

$$\alpha = 3, \quad \beta_m = (n/n+2)^n \beta_n = 1/3 \tag{77}$$

it may be seen in Fig. 7, that an overall very good fit results for $1/m+1/n < 0.5$, say. The corresponding hardness formula

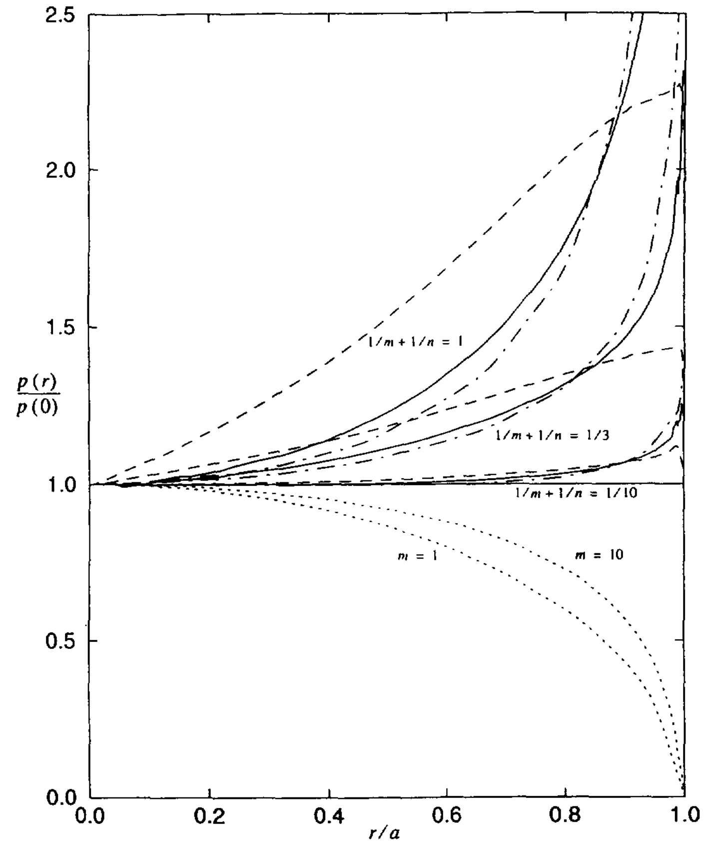

Fig. 8. Normalized contact pressure distribution at spherical indentation of viscoplastic materials for $1/m+1/n = 1$, 1/3, 1/10. (—) General viscoplasticity ($m = n$) present results, (---) plastic flow theory ($n \to \infty$) Biwa and Storåkers (1995), (-··-) creep theory ($m \to \infty$) Storåkers and Larsson (1994), (···) deformation theory of plasticity ($n \to \infty$) Hill *et al.* (1989).

$$\frac{L}{\pi a^2} = \frac{3(n+2)}{n}\sigma_0\left(\frac{a}{3D}\right)^{1/m}\left(\frac{\dot{a}}{3D}\right)^{1/n} \tag{78}$$

is then believed to be practically useful in general circumstances. Based on empirical findings Tabor (1951) proposed for strain-hardening plasticity, in his now celebrated formula, that the present parameters may be adopted as constants and specifically here $\alpha = 2.8$ and $\beta_m = 0.4$.

Finally, some typical pressure distributions are shown in Fig. 8. Also in this case it may be seen that the results are essentially governed by $1/m+1/n$, at least at smaller values, when employing the full constitutive equation. Some results based on the deformation theory of plasticity by Hill *et al.* (1989) are also shown but are at variance reducing to almost Hertzian distributions. However, as was observed in Fig. 7, the difference in mean pressure is much less divergent.

The only real computational difficulty not considered in detail here from a fundamental point of view concerns determination of the shape of the contact contour corresponding to eqn (24) above in genuinely three-dimensional situations. Thus the contour must be determined as to satisfy eqns (32) and (33) above. As in the iteration procedure used already it would seem natural to start with an additional initial conjecture based on the indenter cross section and the corresponding axisymmetric results say, and then use (32) for an approximate velocity field in order to successively refine the contour so that (33) is finally satisfied. No explicit computational algorithm is proposed here but left open for further exploration.

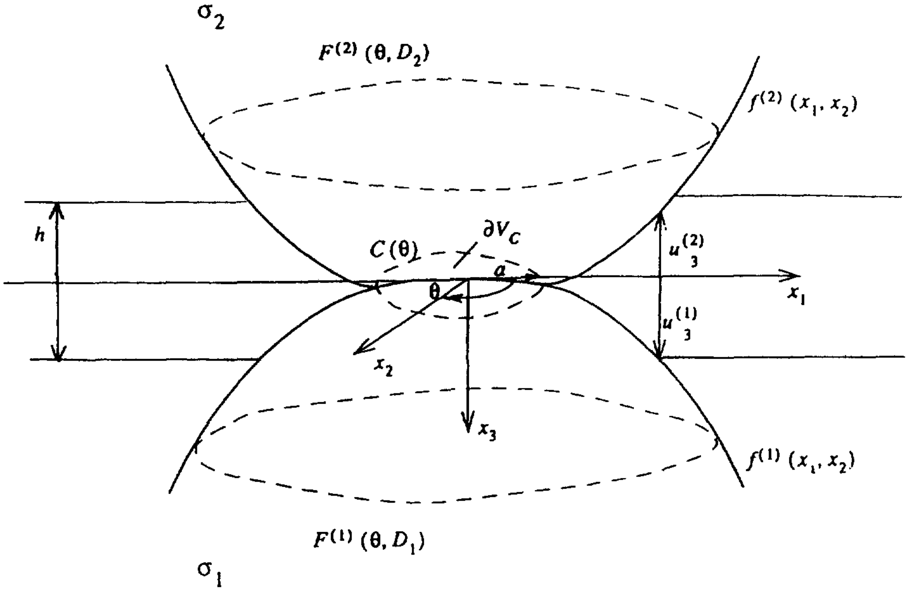

Fig. 9. Contact at two solids, $k = 1, 2$, with material constants σ_k and profile parameters $F^{(k)}(\theta)$, D_k, respectively.

5. GENERAL CONTACT BETWEEN SEVERAL BODIES

The analysis displayed so far has been confined to the problem of a curved rigid die in contact with a deformable half-space for the primary reason to bring out the basic ideas as simply as possible. Although the explicit results discussed are of immediate interest as regards indentation testing, it was forecast that more general results could be accomplished by a simple generalization. Thus in case of two or more deformable bodies interacting, such as the asperity and compaction problems sketched in Fig. 1, no fundamental difficulties will arise when applying the present framework to solve also such combined cases. A compaction problem has recently been analysed by Larsson *et al.* (1996) for an arbitrary number of contacts but as the adopted Hertzian approach is a local one it suffices here to discuss the interaction between two bodies as has been earlier well established in linear elasticity.

Relating to the background above it is assumed that the two bodies have the same constitutive potential structure and homogeneity properties, i.e., m and n, and also that their profiles have the shape exponent p in common save for the single case when one body is flat, $p \to \infty$, and the other arbitrary as above. Otherwise there are no restrictions concerning material constants and surface properties.

With this setting, the field equations will be essentially unaffected and it appears natural to assume ad hoc, with variables introduced in Fig. 9, that

$$\sigma_{ij}^{(k)}(x_l) = \sigma_{ij}^{(0)}(x_l) \quad k = 1, 2 \tag{79}$$

where $\sigma_{ij}^{(0)}$ is the fundamental half-space solution and accordingly continuity of traction at the contour as well as equilibrium is fulfilled.

Then, by scaling displacements correspondingly as

$$u_i^{(k)}(x_l) = (\sigma_0/\sigma_k)^q u_i^{(0)}(x_j) \quad k = 1, 2 \tag{80}$$

where $1/q = 1/m + 1/n$, then the complete field equations are individually satisfied.

It remains to satisfy the nonhomogeneous boundary condition, eqn (18), which now reads

$$u_3^{(1)}(x_1,x_2,0)+u_3^{(2)}(x_1,x_2,0) = h-f^{(1)}(x_1,x_2)-f^{(2)}(x_1,x_2), \quad (x_1,x_2)\in\partial V_c. \tag{81}$$

Thus, by introducing (80) in (81) and choosing

$$\frac{1}{\sigma_0^q} = \frac{1}{\sigma_1^q}+\frac{1}{\sigma_2^q} \tag{82}$$

and

$$f(x_1,x_2) = f^{(1)}(x_1,x_2)+f^{(2)}(x_1,x_2) \tag{83}$$

then (81) reduces to (18) and consequently the original problem of a half-space is recovered from the conjectured fields in the two-body problem.

As regards the contact between nontrivial contours manifested by $C(\theta)$, it is necessary to first determine the eigenfunction $c^p(\theta)$ at indentation of two deformable solids. Again, the procedure will be unaffected as referred to the fundamental problem if in eqn (31) $F(\theta)/D^{p-1}$ is replaced by

$$\frac{F(\theta)}{D^{p-1}} = \frac{F^{(1)}(\theta)}{D_1^{p-1}}+\frac{F^{(2)}(\theta)}{D_2^{p-1}} \tag{84}$$

with the notation as in Fig. 9. It may be pointed out in relation to eqn (84) that there are no restrictions that the solids should be convex once the contact is local and smooth.

Thus with the generalization briefly explained here, the fundamental relations between contact depth and size as in eqn (31) and between mean pressure and the contact radius, eqn (70), now run as before. Results for the two-body problem may then be readily extracted from the fundamental half-space solution. Some explicit applications for spheres of different rigidities and sizes related to composite compaction problems have recently been investigated by Storåkers (1996).

6. CONCLUDING REMARKS

Some general features of self-similarity at contact of linear and nonlinear solids were introduced and discussed in some detail. Thus in cases when both material properties and surface profiles are expressed by aid of homogeneous functions it was shown that the solution to associated problems can be scaled by simple transformation laws. This proved advantageous both from analytical and numerical points of view as the scaling clarifies the dependence of solutions on governing parameters and time in a readily applicable form and reduces an original problem of a moving boundary to a stationary one. From a computational point of view this is favourable and it was further shown that by relying upon an intermediate reduced problem of flat contact the original problem may be simply solved by cumulative superposition. The only price to be paid when reducing the problem to a stationary one is that when material history dependence prevails it will be replaced by nonlocality of the constitutive behaviour. This proved to be no main obstacle, however, as an efficient computational algorithm was devised and its virtues proven by explicit solutions based on general viscoplastic theory. Instead it is believed that the fundamental framework proposed will favourably facilitate procedures at treatment of a wide variety of contact problems in practice and may apply to two-dimensional situations as well with only a slight modification.

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